A VOLUME FORM ON THE KHOVANOV INVARIANT

JUAN ORTIZ-NAVARRO

ABSTRACT. The Reidemeister torsion construction can be applied to the chain complex used to compute the Khovanov homology of a knot or a link. This defines a volume form on Khovanov homology. The volume form transforms correctly under Reidemeister moves to give an invariant volume on the Khovanov homology. In this paper, its construction and invariance under these moves is demonstrated. Also, some examples of the invariant are presented for particular choices for the bases of homology groups to obtain a numerical invariant of knots and links. In these examples, the algebraic torsion seen in the Khovanov chain complex when homology is computed over $\mathbb{Z}$ is recovered.

1. INTRODUCTION

In the 1930s, W. Franz [4] and K. Reidemeister [10] introduced the theory of torsion (also called R-torsion) of a cellular complex in their study of lens spaces. The lens spaces $L(p,q)$ for p fixed have the same homology groups but they are not all homeomorphic. In some cases they are not even homotopy equivalent. The Reidemeister torsion captures some of the interactions that happen under the radar and helps distinguish between many of these spaces. Since Khovanov Homology (cohomology) is defined on a cochain complex, we will talk about Reidemeister torsion on cochain complexes for the remainder of this paper.

Reidemeister Torsion is defined for cochain complexes over a field $\mathbb{F}$ (or, more general, over an asociative ring with multiplicative identity). Over a field, cochain groups become vector spaces. Given an n dimensional vector space V, a multilinear function $T : V^k \to \mathbb{F}$ is called a *k-tensor*. The set of all alternating k-tensors, denoted $\Lambda^k(V)$, is a vector space over $\mathbb{F}$ of dimension $\binom{n}{k}$. A nonzero element of $\Lambda^n(V)$ is called a *volume form* for the space V.

Reidemeister torsion for a cochain complex C defines a volume form on the space:

$$(1) \qquad (C^0)^* \oplus C^1 \oplus (C^2)^* \oplus \ldots C^m (\text{or } (C^m)^*)$$

where the last component of this sum (whether we add the vector or its dual) depends upon the parity of m.

Each basis for a vector space yields a volume form in such a way that if we change basis, the volume form is transformed by the determinant of the change of basis matrix between the two bases. Having a volume form on the space described above and denoting cohomology groups by H^r, one can define a volume form on the space:

$$(2) \qquad (H^0)^* \oplus H^1 \oplus (H^2)^* \oplus \ldots H^m (\text{or } (H^m)^*)$$

In this sense, for acyclic cochain complexes, Reidemeister torsion is a volume form on the 0 vector space. In this case $\Lambda^0(0) = \mathbb{F}$, i.e., we obtain an element of $\mathbb{F}$ that depends only on the bases specified for the chain groups. For non-acyclic cochain

1

complexes, the torsion depends on the bases for the cochain groups as well as the bases specified for the homology groups.

In 2001, Mikhail Khovanov [8] presented a new theory, which assigns homology groups to a knot diagram. His theory sparked new questions in knot theory, as well as, proposed new ways of attacking old problems in topology. Khovanov theory constructs a complex from a diagram of a link. The cohomology groups obtained from this complex are invariant under Reidemeister moves in the diagram, which makes them a topological invariant of the link or knot under study. Cochain groups in this complex are made of tensor products of a graded algebra over a ring and coboundary operators arise from operations in the algebra. The complex itself decomposes as subcomplexes that preserve the grading in the algebra. From this point of view, Khovanov homology recovers other invariants of links, such as the well-known Jones polynomial introduced by Jones in [6].

In this paper, we construct a topological invariant for knots and links using Reidemeister torsion, which gives a well defined volume form for the Khovanov homology. We showed this by demonstrating the volume form is preserved under Reidemeister moves by looking at the cochain maps inducing isomorphisms in homology for each of this moves, as presented in [8]. We also demonstrate that, for acylic subcomplexes, the Reidemeister torsion gives an invariant number for knots and links.

In Section 2, we present a brief introduction to Reidemeister torsion. Section 3 serves as an introduction to Khovanov homology. In Section 4, we demonstrate the volume form on Khovanov homology. In Section 5, we calculate the form for a simple example and show tables for some knots and links for a very special choice of bases for the cohomology groups.

2. R-torsion

As defined in [14].

2.1. Definition for acyclic complexes.
Let $\mathbb{F}$ be a field and D be a finite-dimensional vector space over $\mathbb{F}$. Suppose that $\dim D = k$ and pick two (ordered) bases $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$ of D. Let

$$b_j = \sum_{i=1}^{k} a_{ij} c_i, \qquad j = 1, \ldots, k,$$

The matrix $(a_{ij})_{i,j=1,\ldots,k}$ is called the *transition matrix* between the basis b and c and it is a nondegenerate $(k \times k)$ - matrix over $\mathbb{F}$. We write

$$[b/c] = \det(a_{ij}) \ \in \mathbb{F}^* = \mathbb{F} - 0$$

One can show that the relation, $b \sim c$ if and only if $[b/c] = 1$, is an equivalence relation.

Let C, D and E be vector spaces over $\mathbb{F}$. Let

$$0 \to C \xrightarrow{i} D \xrightarrow{\pi} E \to 0$$

be a short exact sequence of vector spaces. Then $\dim D = \dim C + \dim E$. Let $c = (c_1, \ldots, c_k)$ be a basis for C, $d = (d_1, \ldots, d_k)$ be a basis for D and $e = (e_1, \ldots, e_l)$

be a basis for E. Since β is surjective, we may lift each e_i to some $\tilde{e}_i \in D$, such that $\beta(\tilde{e}_i) = e_i$. We will call $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_l)$ a *pullback* for e. We set

$$c\,\tilde{e} = (c_1, \ldots, c_k, \tilde{e}_1, \ldots, \tilde{e}_l).$$

Then $c\,\tilde{e}$ is a basis for D.

Let

$$C = (0 \to C^0 \xrightarrow{\partial^0} C^1 \to \cdots \xrightarrow{\partial^{m-2}} C^{m-1} \xrightarrow{\partial^{m-1}} C^m \to 0)$$

be an acylcic based cochain complex over $\mathbb{F}$. Set $B^{r-1} = \mathrm{Im}(\partial^{r-1} : C^{r-1} \to C^r) \subset C^r$. Since C is acyclic,

$$C^r/B^{r-1} = C^r/\ker(\partial^r : C^r \to C^{r+1}) \cong im\,\partial^r = B^r.$$

In other words, the sequence

$$0 \to B^{r-1} \hookrightarrow C^r \xrightarrow{\partial^r} B^r \to 0$$

is exact. Choose a basis b^r of B^r for $r = -1, \ldots, m$. By the above construction, $b^{r-1}\,\tilde{b}^r$ is a basis of C^r, which can be compared with the basis c^r of C^r.

Definition 1. *The Reidemeister torsion of C is*

$$\tau(C) = \left| \prod_{r=0}^{m} [b^{r-1}\,\tilde{b}^r/c^r]^{(-1)^{r+1}} \right| \in \mathbb{F}^*.$$

Remarks: (see [14])
- $\tau(C)$ does NOT depend on the choice for b^r and its pullback $\tilde{b}^r$.
- $\tau(C)$ DOES depend on c^r, which is called the *distinguished basis* for the cochain group C^r.
- If another basis for the cochain groups is equivalent to the distinguished basis then $\tau(C)$ is the same for both bases. Indeed, if C' is the same acyclic cochain complex C based $c' = (c'^0, \ldots, c'^m)$, then

$$\tau(C') = \tau(C) \prod_{i=0}^{m} [c^i/c'^i]^{(-1)^{i+1}}.$$

2.2. **R-torsion as a volume form.** R-torsion can be understood as a volume form on the space:

(3) $$(C^0)^* \oplus (C^1) \oplus (C^2)^* \oplus \cdots \oplus (C^m) \;(\text{or } (C^m)^*)$$

where $(C^r)^* = Hom(C^r, \mathbb{F})$ is the dual of C^r, and the last component (C^m) (or its dual) depends on the parity of m.

Consider the cochain complex (D, Δ) (not necessarily acyclic), then by the 1st isomorphism theorem:

$$0 \to \ker \Delta^i \hookrightarrow D^i \xrightarrow{\Delta^i} \mathrm{Im}\,\Delta^i \to 0$$

is exact. Moreover,

$$0 \to B^{i-1} \hookrightarrow \ker \Delta^i \xrightarrow{\pi} H^i \to 0$$

is also exact. Thus, having basis b^i for $B^i = \mathrm{Im}\,\Delta^i$ (as before) and $[h^i]$ for H^i for every i, we obtain the basis $[b^{i-1}\,h^i]$ for $\ker \Delta^i$, where $\pi(h^i) = [h^i]$. Furthermore, $[b^{i-1}\,h^i\,\tilde{b}^i]$ is a basis for D^i.

Definition 2. *The Reidemeister torsion of D is*

$$\tau(D) = \left| \prod_{r=0}^{m} [b^{r-1} \, h^r \, \tilde{b}^r / c^r]^{(-1)^{r+1}} \right| \in \mathbb{F}^*.$$

Remarks: (see [14])
- $\tau(C)$ does NOT depend on the choice for b^r and its pullback $\tilde{b}^r$.
- Using the quotient map $\ker \Delta^i \xrightarrow{\pi} H^i$, and a pullback for a cohomology basis, one can think of cohomology as lying in the cochain group and the restriction of the volume form on cohomology makes R-torsion a volume form for the vector space $(H^0)^* \oplus H^1 \oplus (H^2)^* \oplus \ldots H^m$ (or $(H^m)^*$), (depending on the parity of m).

2.3. **Mapping Cone.** Suppose there is a cochain map $\varphi : (C, \partial_C) \to (D, \partial_D)$ between two cochain complexes over a field.

Definition 3. *The mapping cone $m(\varphi)$ is a cochain complex (E, δ) with cochain groups:*

$$E^r = C^r \oplus D^{r-1}$$

and coboundary operator:

$$\delta^r : E^r \to E^{r+1}$$

so that for $c \in C^r$ and $d \in D^{r-1}$

$$\delta^r(c, d) = \left(\partial_C^r(c), \partial_D^{r-1}(d) + (-1)^r \varphi_r(c) \right)$$

or

$$\delta^r = \begin{pmatrix} \partial_C^r & 0 \\ (-1)^r \varphi^r & \partial_D^{r-1} \end{pmatrix}$$

where this matrix acts on a vector whose first block represents an element in C^r and the second block represents an element in D^{r-1}. (Similar notation should be understood similarly through the rest of this paper.)

Note that the distinguished bases for the cochain groups of C and D combine to give distinguished bases for the cochain groups of $m(\varphi)$.

2.4. **Quasi-isomorphisms.** A cochain map $\varphi : (C, \partial_C) \to (D, \partial_D)$ between two cochain complexes over a field induces a well defined map in cohomology, $\varphi_*^r : H^r(C) \to H^r(D)$,

$$\varphi_*^r([h]) = [\varphi^r(h)] \qquad \text{for} \quad [h] \in H^r(C)$$

If this map is an isomorphism for every r, then the cochain map is called a *quasi-isomorphism*. Note that the mapping cone of a quasi-isomorphism is acyclic. The short exact sequence:

$$0 \to D^{r-1} \hookrightarrow D^{r-1} \oplus C^r \xrightarrow{\pi} C^r \to 0$$

induces a long exact sequence in cohomology:

$$\cdots \to H^{r-1}(D) \to H^r(E) \to H^r(C) \xrightarrow{\varphi_*^r} H^r(D) \to H^{r+1}(E) \to \cdots$$

Since $\varphi_*^r : H^r(C) \cong H^r(D)$, one can decompose the sequence to obtain:

$$0 \to H^r(E) \to 0$$

to get $H^r(E) = 0$ for every r.

In [3], Chung and Lin presented a theory for the torsion of quasi-isomorphisms.

Definition 4. *The torsion $\tau(\varphi)$ of the quasi-ismorphism $\varphi : (C, \partial_C) \to (D, \partial_D)$ is*

$$(4) \qquad \tau(\varphi) = \left| \prod_{r=0}^{m} \frac{[b^{r-1}\, h^r\, \tilde{b}^r / c^r]}{[b'^{r-1}\, \varphi_*^r(h^r)\, \tilde{b'}^r / c^r]}^{(-1)^{r+1}} \right|$$

where b^i and b'^i are bases for $\operatorname{Im} \partial_C$ and $\operatorname{Im} \partial_D$ respectively.

Remarks: (see [3])
- $\tau(\varphi)$ does NOT depend on the choice of basis for the cohomological groups.
- Their definition coincides with the torsion of the acyclic mapping cone of the quasi-isomorphism and is independent of the choice of bases for the cohomology groups.

2.5. **A special quasi-isomorphism.** Consider a quasi-isomorphism $\varphi : (C, \partial_C) \to (D, \partial_D)$ such that its matrix representation according to the distinguished bases for all cochain groups for C and D is given by identity matrices (not necessarily the identity map, since D and C may not have the same cochain groups). Then, the coboundary operator δ^r of its mapping cone looks like:

$$(5) \qquad \delta^r = \begin{pmatrix} \partial_C^r & 0 \\ (-1)^r Id & \partial_D^{r-1} \end{pmatrix}$$

Now:

$$\operatorname{Im} \delta^r = \operatorname{Im} \partial_C^r \bigoplus D^r$$

Then $\mathcal{B}^r = (b_C^r, 0) \cup (0, d^r)$ is a basis for $\operatorname{Im} \delta^r$, where b_C^r is a basis for $\operatorname{Im} \partial_C^r$ and d^r is the distinguished basis for D^r. Then a pullback is given by $\tilde{\mathcal{B}}^r = c^r$, where c^r is the distinguished basis for C^r. Therefore:

$$(6) \qquad [\mathcal{B}^{r-1}\, \tilde{\mathcal{B}}^r / c^r \oplus d^{r-1}] = \begin{vmatrix} * & Id \\ Id & 0 \end{vmatrix} = 1$$

Thus the Reidemeister torsion of the quasi-isomorphism is 1. Note that in this case, using the isomorphism φ_* and its dual φ_*^*:

$$(7)$$
$$(H_C^0)^* \oplus H_C^1 \oplus (H_C^2)^* \oplus \ldots H_C^m (\text{or } (H_C^m)^*) \cong (H_D^0)^* \oplus H_D^1 \oplus (H_D^2)^* \oplus \ldots H_D^m (\text{or } (H_D^m)^*)$$

and by (4), the volume form is preserved on this space generated from the cohomology groups.

3. The Khovanov Chain Complex

As defined in [1].

All links are oriented in an oriented Euclidean space. We will present them using projections to the plane. Let D be a diagram of a link L, $\mathcal{X}$ be the set of crossings of D, $n = |\mathcal{X}|$. Let us number the elements of $\mathcal{X}$ from 1 to n and write $n = n_+ + n_-$ where n_+ (n_-) is the number of right-handed (left-handed) crossings in $\mathcal{X}$.

3.1. **Spaces.**

Definition 5. *Let $W = \bigoplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$. The graded dimension of W is the power series $\operatorname{qdim} W := \sum_m q^m \dim W_m$.*

Definition 6. *If $W = \bigoplus_m W_m$ is a graded vector space, we set $W\{l\}_m := W_{m-l}$, so that $\operatorname{qdim} W\{l\} = q^l \operatorname{qdim} W$.*

Definition 7. *If $\bar{C}$ is a cochain complex $\ldots \to \bar{C}^r \xrightarrow{d^r} \bar{C}^{r+1} \ldots$ of vector spaces, and if $C = \bar{C}[s]$, then $C^r = \bar{C}^{r-s}$ (differentials also shifted).*

Let V be the graded vector space with two basis elements x and 1 whose degrees are -1 and 1 respectively, so that $qdimV = q + q^{-1}$. For every vertex $\alpha \in \{0, 1\}^{\mathcal{X}}$ of the cube $\{0, 1\}^{\mathcal{X}}$, we associate the graded vector space $V_\alpha(D) := V^{\otimes k}\{r\}$, where k is the number of cycles in the smoothing of D corresponding to α and r is the height $|\alpha| = \sum_i \alpha_i$ of α. We set the rth cochain group $\|D\|^r$ (for $0 \le r \le n$) to be the direct sum of all the vector spaces at height r: $\|L\|^r := \bigoplus_{\alpha : r = |\alpha|} V_\alpha(D)$. Finally, we set $C(D) := \|D\|[-n_-]\{n_+ - 2n_-\}$.
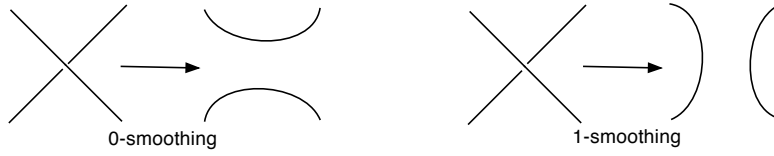

FIGURE 1. Smoothing a crossing

3.2. **Distinguished Basis.** Each cochain group in the Khovanov complex has an ordered basis in the following way. The vector space V has $\{1, x\}$ as a basis. Now, consider the tensor product of r copies of V. Using reverse lexicographic order, we get the ordered basis:

$$
\begin{aligned}
&1 \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \\
&x \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \\
&1 \otimes x \otimes 1 \otimes \ldots \otimes 1 \\
&x \otimes x \otimes 1 \otimes \ldots \otimes 1 \\
&\qquad\qquad \vdots \\
&1 \otimes x \otimes x \otimes \ldots \otimes x \\
&x \otimes x \otimes x \otimes \ldots \otimes x
\end{aligned}
$$

(8)

For every vertex α, $V_\alpha(D)$ is assigned such a basis, so that the cochain group $\|D\|^r$ has basis consisting of the injection of each of these bases. This basis will be the distinguished basis for the rth cochain group.

3.3. **Maps.** The space $V_\alpha(D)$ on each vertex α has as many tensor factors as there are components in the smoothing S_α. Thus, we put these tensor factors in V_α and cycles in S_α in bijective correspondence. Each edge ξ of the cube maps the vector spaces at its ends. For any edge ξ, the smoothing at the tail of ξ differs from the smoothing at the head of ξ slightly: either two of the components merge into one or one of the components splits in two. So for any ξ, we set d_ξ to be the identity on the tensor factors corresponding to the components that don't participate, and then we complete the definition of d_ξ using two linear maps: $m : V \otimes V \to V$ and

$\Delta : V \to V \otimes V$ defined as follows:

(9) $\qquad \left(V \otimes V \xrightarrow{m} V \right) \qquad\qquad m : \begin{cases} 1 \otimes x \mapsto x & 1 \otimes 1 \mapsto 1 \\ x \otimes 1 \mapsto x & x \otimes x \mapsto 0 \end{cases}$

(10) $\qquad \left(V \xrightarrow{\Delta} V \otimes V \right) \qquad\qquad \Delta : \begin{cases} 1 \mapsto 1 \otimes x + x \otimes 1 \\ x \mapsto x \otimes x \end{cases}$

The height $|\xi|$ of an edge ξ is defined to be the height of its tail. Hence, if the maps on the edges are called ∂_ξ, then the vertical collapse of the cube to a cochain complex becomes $\partial^r := \sum_{|\xi|=r} (-1)^\xi \partial_\xi$, where $(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$, and j is the location where the smoothings at the head and tail of the edge differ.

3.4. Khovanov Invariant.
Let $\mathcal{H}^r(D)$ denote the rth cohomology group of $C(D)$. It is a graded vector space. Let

$$Kh(L) := \sum_r t^r \mathrm{qdim} \mathcal{H}^r(D)$$

Theorem 1. $\mathcal{H}^r(L)$ *is an invariant for the link L and $Kh(L)$ is a link invariant that specializes to the Jones polynomial at $t = -1$.*

see [8].

4. The volume form

Theorem 2. *The Khovanov homology has a volume form which is invariant for knots and links.*

In this (long) section, invariance of the volume form is demonstrated under each of the Reidemeister moves. To obtain this, it will be shown that for each of the moves the mapping cone of the quasi-isomorphisms relating each side of the move has Reidemeister Torsion 1.

As in [9]. $\overline{\mathcal{C}}(D)$ will denote the complex assign to a diagram of a link prior to any shifts, i.e.,

$$\mathcal{C}(D) = \overline{\mathcal{C}}(D)[-n_-]\{n_+ - 2n_-\}$$

4.1. Reidemeister I.
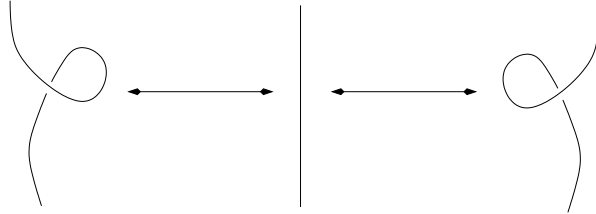The Reidemeister I move has two variants.

FIGURE 2. Reidemeister I move

The left side (the negative kink) can be obtained by a series of Reidemeister I (positive kink), Reidemeister II and Reidemeister III moves by the Whitney trick. Hence, invariance for negative kinks follows from the Reidemeister I (positive), Reidemeister II, and Reidemeister III moves.
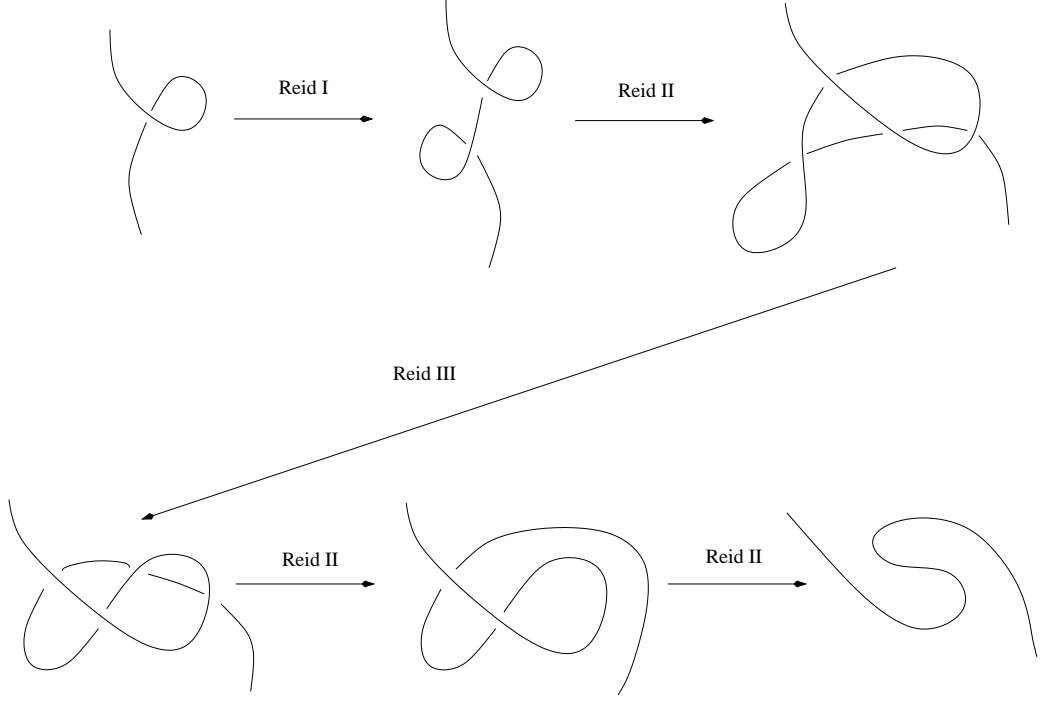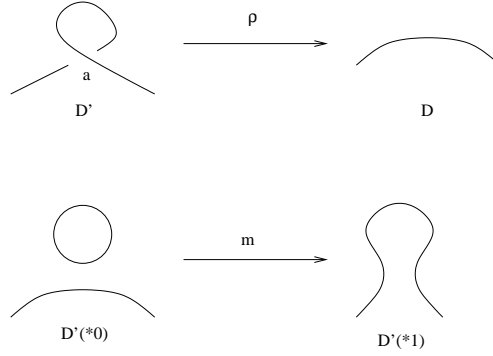
FIGURE 3. Whitney trick

FIGURE 4. Reidemeister I Invariance

Let a be the crossing which appears only in D'. The set $\mathcal{I}'$ of crossings of D' is $\mathcal{I}$, the set of crossings of D, followed by a as an ordered $|\mathcal{I}'|$-tuple. Let $D'(*0)$ and $D'(*1)$ denote D' with only its last crossing (that is a) resolved to its 0- and 1-resolutions, respectively.

Consider $\partial'_{0\to1} : \overline{\mathcal{C}}(D'(*0)) \to \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$, which is always multiplication and define
$$X_1 = \operatorname{Ker} \partial'_{0\to1}$$
and
$$X_2 = \{y \otimes 1 + z \,|\, y \in \overline{\mathcal{C}}(D), z \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}\} \ .$$

X_1 and X_2 are subcomplexes of $\overline{\mathcal{C}}(D')$. $\overline{\mathcal{C}}(D')$ decomposes as $X_1 \oplus X_2$ as a chain complex. (see [8])

Now, consider the r-th cochain group of the complex $\overline{\mathcal{C}}(D')^r$. This space has a distinguished basis coming from the construction of the complex as explained in Section (3).

As a basis for $X_1^r = \operatorname{Ker} \partial'^r_{0 \to 1}$, we use $\{\dots \otimes 1 \otimes x - \dots \otimes x \otimes 1, \dots \otimes x \otimes x\}$ and, for X_2^r, we use the subbasis of the distinguished basis consisting of all basis elements with a 1 in the last tensor for components corresponding to states with a positive smoothing at the crossing a and all the basis elements for components corresponding to states with a negative smoothing at the crossing a.

Using this ordered basis, the change of basis matrix between the basis coming from X_1^r and X_2^r and the distinguished basis for $\overline{\mathcal{C}}(D')^r$ consists of diagonal blocks of the form:

(11)
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

for states with a positive marker at the crossing a. The states with a negative marker at this crossing belong to the subcomplex X_2 and the basis is unchanged. In any case, the bases for the cochain groups coming from the decomposition of the complex as subcomplexes X_1 and X_2 are equivalent to the distinguished bases.

The subcomplex $X_2 = \{y \otimes 1 + z | y \in \overline{\mathcal{C}}(D), z \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}\}$ is acyclic. (see [8]) Let's find its Reidemeister torsion. We only need to find basis for the image of the boundary operator and a pullback for this basis.
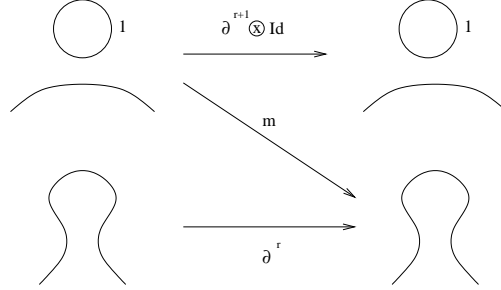

FIGURE 5. $\partial'^r|_{X_2^r} : X_2^r \to X_2^{r+1}$

The image of $\partial'^r|_{X_2^r}$ consists of:

$$\{(\partial^{r+1}(y) \otimes 1, y), \, y \in \overline{\mathcal{C}}(D)^{r+1}\}$$

Thus, a basis b'^r for $\operatorname{Im} \partial'^r$ is given by:

$$(b^{r+1} \otimes 1, c^{r+1})$$

where b^{r+1} is a basis for ∂^{r+1} on $(\overline{\mathcal{C}}(D), \partial)$ and c^{r+1} is the distinguished basis for $\overline{\mathcal{C}}(D(*1))^{r+1}$.

A pullback $\tilde{b}'^r$ for this basis is $(c^{r+1} \otimes 1, 0)$. Hence $[b'^{r-1}\tilde{b}'^r/\mathcal{X}_2^r]$ is the determinant of the change of basis matrix between the ordered basis for $b'^{r-1} \tilde{b}'^r$ as described,

and $\mathcal{X}_2^r$, the previously described basis for X_2^r:

$$(12) \qquad \begin{pmatrix} b^r \otimes 1 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

which has determinant ± 1.

The map

$$\rho : \qquad X_1 \qquad \longrightarrow \overline{\mathcal{C}}(D)\{1\}$$
$$y \otimes 1 + z \otimes \mathbf{x} \quad \longmapsto \quad z$$

induces an isomorphism between $\mathcal{H}(D')$ and $\mathcal{H}(D)$. (see [8]) With the described bases, the matrix representation of the cochain map is the identity matrix for all cochain groups. Hence by (6) the Reidemeister torsion of the quasi-isomorphism is 1.
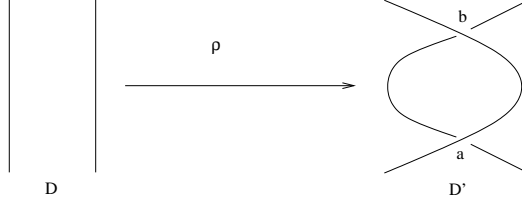

FIGURE 6. Reidemeister II invariance

4.2. **Reidemeister II Move.** As before, the set $\mathcal{I}'$ of crossings of D' is $\mathcal{I}$, the set of crossings of D, followed by a, then b as an ordered $|\mathcal{I}'|$-tuple.

Let:

$$\begin{aligned} X_1 &= \{z + \alpha(z) | z \in \overline{\mathcal{C}}(D'(*01))[-1]\{-1\}\} \\ X_2 &= \{z + \partial' y | z, y \in \overline{\mathcal{C}}(D'(*00))\} \\ X_3 &= \{z + y \otimes 1 | z, y \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}\} \end{aligned}$$

where $\alpha(z) = -\partial'_{01 \to 11}(z) \otimes 1 \in \overline{\mathcal{C}}(D'(*10))[-1]\{-1\}$.

Then, $\overline{\mathcal{C}}(D')$ is the direct sum of its subcomplexes X_1, X_2, and X_3. (see [8])

Now consider the r-th cochain group of the complex for the left side , $\overline{\mathcal{C}}(D')^r$. This space has a distinguished basis coming from the construction of the complex. Let's see how the decomposition of the complex into these subcomplexes affects the torsion. Let's consider first the case when $\partial'_{01 \to 11} = m$. We can handle this locally since everywhere else the complex is unchanged.

Since $X_1^r = \{z + \alpha(z) | z \in \overline{\mathcal{C}}(D'(*01))[-1]\{-1\}^r\}$, we can choose the basis:

$$\begin{aligned} (1 \otimes 1 &\quad , \quad 1 \otimes 1) \\ (x \otimes 1 &\quad , \quad x \otimes 1) \\ (x \otimes 1 &\quad , \quad 1 \otimes x) \\ (0 &\quad , \quad x \otimes x) \end{aligned}$$

where the first component is an element of $\overline{\mathcal{C}}(D'(*10))^r$ and the second component lies on $\overline{\mathcal{C}}(D'(*01))^r$.
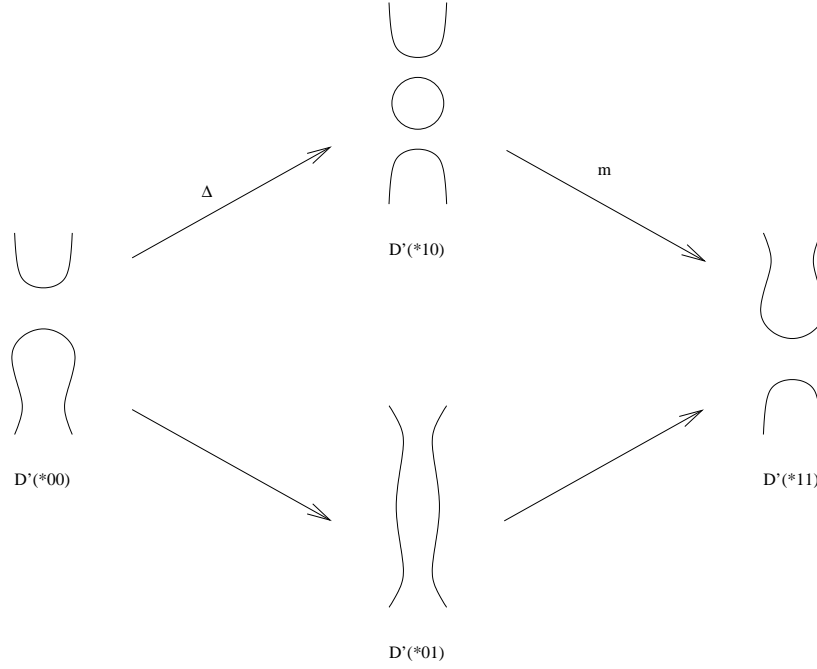
FIGURE 7. Complex for the right side $\overline{\mathcal{C}}(D')$

Since $X_2^r = \{z + \partial'y | z \in \overline{\mathcal{C}}(D'(*00))^r, y \in \overline{\mathcal{C}}(D'(*00))^{r-1}\}$, a basis for the z part (which lies in $\overline{\mathcal{C}}(D'(*00))^r$), is just $\{1, x\}$, hence, unchanged, and for the $\partial'(y)$ part the basis is

$$\{(1 \otimes x + x \otimes 1, 1 \otimes x + x \otimes 1), (x \otimes x, x \otimes x)\}$$

which lies on $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$.

Now, $X_3^r = \{z + y \otimes 1 | z \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}^r, y \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}^{r+1}\}$ has as basis $\{1, x\}$ on $\overline{\mathcal{C}}(D'(*11))^r$ and

$$\{(1 \otimes 1, 0), (x \otimes 1, 0)\}$$

on $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$.

Thus, the bases for $\overline{\mathcal{C}}(D'(*00))^r$ and $\overline{\mathcal{C}}(D'(*11))^r$ are the distinguished bases. And for $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$ the change of basis matrix between the the ordered basis coming from X_1^r, X_2^r, and X_3^r (in the order presented) and the distinguished basis for $\overline{\mathcal{C}}(D')^r$ is:

$$(13) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which has determinant ± 1.
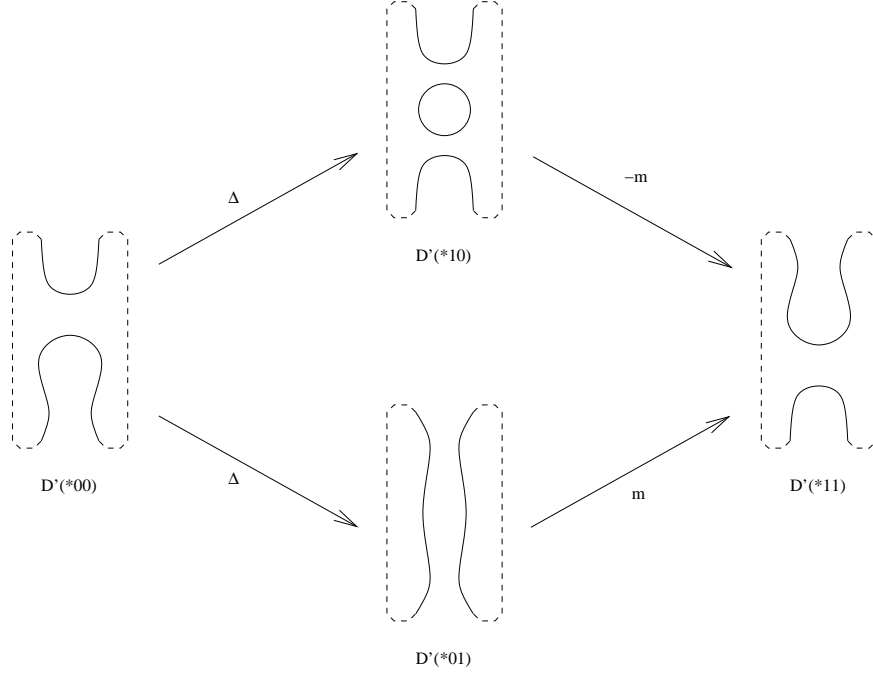
FIGURE 8. $\overline{\mathcal{C}}(D')$ for the case when $\partial'_{01\to11} = m$

Let's consider now the case when $\partial'_{01\to11} = \Delta$. We can handle this locally since everywhere else the complex is unchanged.

Since $X_1^r = \{z + \alpha(z) | z \in \overline{\mathcal{C}}(D'(*01))[-1]\{-1\}^r\}$, we can choose the basis:

$$\{(1 \otimes x \otimes 1 + x \otimes 1 \otimes 1, 1), (x \otimes x \otimes 1, x)\}$$

where the first component is an element of $\overline{\mathcal{C}}(D'(*10))^r$ and the second component lies on $\overline{\mathcal{C}}(D'(*01))^r$.

Since $X_2^r = \{z + \partial' y | z \in \overline{\mathcal{C}}(D'(*00))^r, y \in \overline{\mathcal{C}}(D'(*00))^{r-1}\}$, a basis for the z part (which lies in $\overline{\mathcal{C}}(D'(*00))^r$) is $\{1 \otimes 1, x \otimes 1, 1 \otimes x, x \otimes x\}$, and for the $\partial'(y)$ part is

$$
\begin{aligned}
(1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 &\quad, \quad 1) \\
(x \otimes 1 \otimes x + x \otimes x \otimes 1 &\quad, \quad x) \\
(1 \otimes x \otimes x &\quad, \quad x) \\
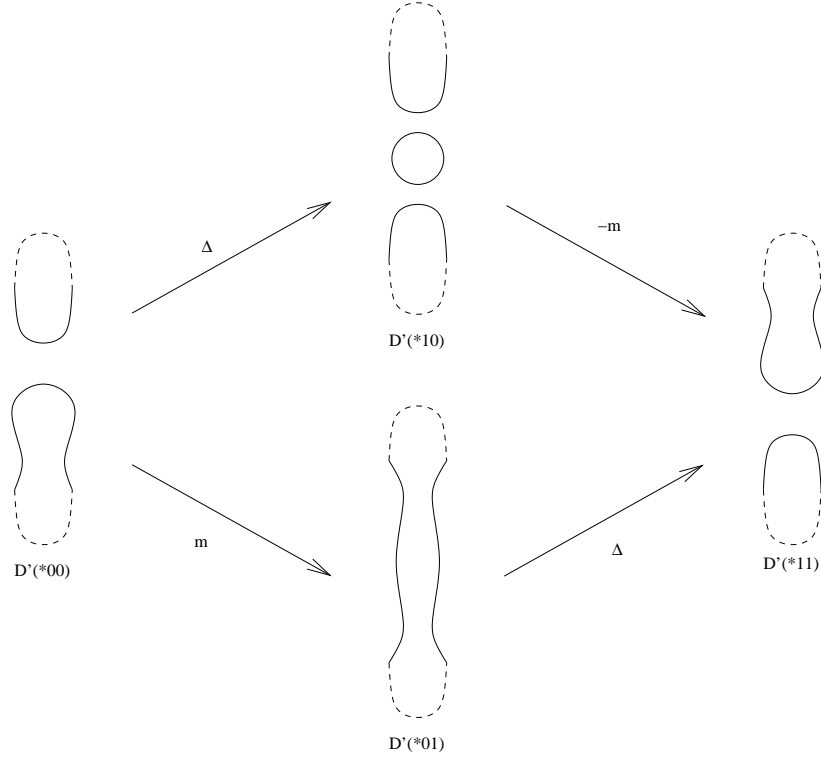(x \otimes x \otimes x &\quad, \quad 0)
\end{aligned}
$$

which lies on $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$.

Now, $X_3^r = \{z + y \otimes 1 | z \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}\}^r, y \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}\}^{r+1}$ has as basis $\{1 \otimes 1, x \otimes 1, 1 \otimes x, x \otimes x\}$ on $\overline{\mathcal{C}}(D'(*11))[-2]\{-2\}^r$ and

$$\{(1 \otimes 1 \otimes 1, 0), (x \otimes 1 \otimes 1, 0), (1 \otimes x \otimes 1, 0), (x \otimes x \otimes 1, 0)\}$$

on $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$.

Thus, the bases for $\overline{\mathcal{C}}(D'(*00))^r$ and $\overline{\mathcal{C}}(D'(*11))^r$ are the distinguished bases. And for $\overline{\mathcal{C}}(D'(*10))^r \oplus \overline{\mathcal{C}}(D'(*01))^r$, the change of basis matrix between the ordered

FIGURE 9. $\overline{\mathcal{C}}(D')$ for the case when $\partial'_{00\to11} = -\Delta$

basis coming from X_1^r, X_2^r, and X_3^r (in the order presented) and the distinguished basis for $\overline{\mathcal{C}}(D')^r$ is:

(14)
$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

which also has determinant ± 1.

Therefore, the bases for the cochain groups coming from the decomposition of the complex as the subcomplexes X_1, X_2, and X_3 are equivalent to the distinguished bases in both cases.

The subcomplex $X_2 = \overline{\mathcal{C}}(D'(*00)) \bigoplus \partial'(\overline{\mathcal{C}}(D'(*00)))$ is acyclic. (see [8]) Let $a \in X_2^r$, then $a = z + \partial'^{r-1}(y)$ where $z \in \overline{\mathcal{C}}(D'(*00))^r$ and $y \in \overline{\mathcal{C}}(D'(*00))^{r-1}$,

hence

$$
\begin{aligned}
\partial'^r(a) &= \partial'^r(z) + \partial'^r(\partial'^{r-1}(y)) \\
&= \Delta(z) + \partial'_{00\to01}(z) + \partial'_{00\to00}(z)
\end{aligned}
$$

Let b'^r be a basis for $\operatorname{Im}\partial'^r|_{X_2^r}$. Note that since Δ has trivial kernel, a pullback for this basis $\tilde{b}'^r$ is the distinguished basis $c'(00)^r$ for $\overline{\mathcal{C}}(D'(*00))^r$.

Hence, $[b'^{r-1}\,\tilde{b}'^r/\mathcal{X}_2^r]$ is the determinant of the change of basis matrix between the ordered basis for $b'^{r-1}\tilde{b}'^r$ as described and $\mathcal{X}_2^r$, the previously described basis for X_2^r. These bases are the same since b'^{r-1} is the basis we had for $\partial'^{r-1}\overline{\mathcal{C}}(D'(*00))^{r-1}$ and $\tilde{b}'^r$ is the basis we had for $\overline{\mathcal{C}}(D'(*00))^r$. Hence, $[b'^{r-1}\,\tilde{b}'^r/\mathcal{X}_2^r] = 1$.

Consider the acyclic (see [8]) subcomplex $X_3 = \{z + y \otimes 1 | z, y \in \overline{\mathcal{C}}(D'(*11))\}$. Let $a \in X_3^r$, then $a = z + y \otimes 1$ where $z \in \overline{\mathcal{C}}(D'(*11))^r, y \in \overline{\mathcal{C}}(D'(*11))^{r+1}$and, hence,

$$
\begin{aligned}
\partial'^r(a) &= \partial'^r(z) + \partial'^r(y \otimes 1) \\
&= \partial'^r_{11\to11}(z) - m(y,1) + \partial'^r_{01\to01}(y \otimes 1) \\
&= \partial'^r_{11\to11}(z) - y + \partial'^r_{01\to01}(y \otimes 1)
\end{aligned}
$$

Let b'^r be a basis for $\operatorname{Im}\partial'^r|_{X_3^r}$. Note that since $m(*,1)$ generates all of $\overline{\mathcal{C}}(D'(*11))^{r+1}$,

$$
\operatorname{Im}\partial'^r|_{X_3^r} = \{(-y, \partial'^r_{01\to01}(y \otimes 1))|y \in bcn(D'(*11))^{r+1}\}
$$

hence, a basis is given by:

$$
b'^r = (-c'(*11)^{r+1}, b'^r_{01})
$$

where $c'(*11)$ is the distinguished basis for $\overline{\mathcal{C}}(D'(*11))^r$ and b'^r_{01} is a basis for $\operatorname{Im}\partial'^r_{01\to01}$. A pullback for this basis is:

$$
\tilde{b}'^r = (0, c'(*11)^{r+1} \otimes 1)
$$

Hence, $[b'^{r-1}\,\tilde{b}'^r/\mathcal{X}_3^r]$ is the determinant of the change of basis matrix between the ordered basis for $b'^{r-1}\,\tilde{b}'^r$ as described, and the basis $\mathcal{X}_3^r$, described previously for X_3^r:

$$
\begin{pmatrix} -Id & 0 \\ b'^{r-1}_{01} & Id \end{pmatrix}
$$

which has determinant ± 1.

Moreover:

$$
\begin{aligned}
\rho: \quad \overline{\mathcal{C}}(D)[-1]\{-1\} &\longrightarrow X_1 \\
z &\longmapsto (-1)^r(z + \alpha(z))
\end{aligned}
$$

induces an isomorphism between $\mathcal{H}(D)$ and $\mathcal{H}(D')$. (see [8]) With the prescribed bases, the matrix representation of the cochain map is the identity matrix. Hence by (6), the Reidemeister torsion of the quasi-isomoporhism is 1.

4.3. **Reidemeister III Move.** Again, a, b, c and a', b', c' are the last three elements in $\mathcal{I}$ and $\mathcal{I}'$ and the others are in the same order.
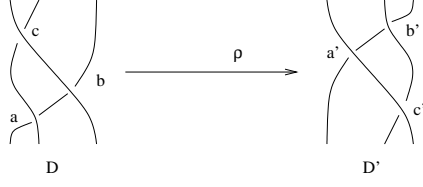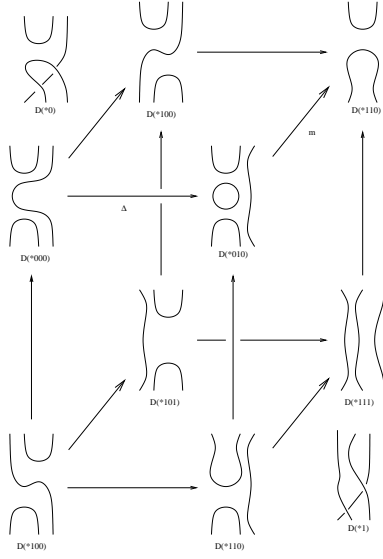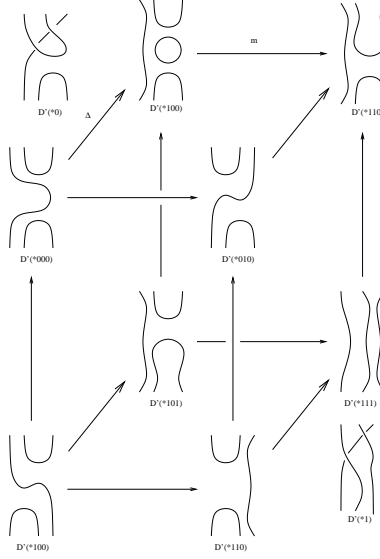
FIGURE 10. Reidemeister III invariance

FIGURE 11. Complex for the left side $\overline{\mathcal{C}}(D)$

Define $\alpha, \beta, \alpha', \beta'$ as

$$
\begin{aligned}
\alpha: \quad & \overline{\mathcal{C}}(D(*110))[-2]\{-2\} && \longrightarrow \overline{\mathcal{C}}(D(*010))[-1]\{-1\} \approx \overline{\mathcal{C}}(D(*110))[-1]\{-1\} \otimes \mathbf{A} \\
& z && \longmapsto z \otimes 1 \\
\beta: \quad & \overline{\mathcal{C}}(D(*100))[-1]\{-1\} && \longrightarrow \overline{\mathcal{C}}(D(*010))[-1]\{-1\} \\
& z && \longmapsto \alpha \partial_{100 \to 110}(z) \\
\alpha': \quad & \overline{\mathcal{C}}(D'(*110))[-2]\{-2\} && \longrightarrow \overline{\mathcal{C}}(D'(*100))[-1]\{-1\} \approx \overline{\mathcal{C}}(D'(*110))[-1]\{-1\} \otimes \mathbf{A} \\
& z && \longmapsto z \otimes 1 \\
\beta': \quad & \overline{\mathcal{C}}(D'(*010))[-1]\{-1\} && \longrightarrow \overline{\mathcal{C}}(D'(*100))[-1]\{-1\} \\
& z && \longmapsto -\alpha' \partial'_{010 \to 110}(z) \ .
\end{aligned}
$$

FIGURE 12. Complex for the right side $\overline{\mathcal{C}}(D')$

$\overline{\mathcal{C}}(D)$ and $\overline{\mathcal{C}}(D')$ can be decomposed into their subcomplexes as below:

$$
\begin{aligned}
\overline{\mathcal{C}}(D) &= X_1 \oplus X_2 \oplus X_3 \\
X_1 &= \{x + \beta(x) + y | x \in \overline{\mathcal{C}}(D(*100))[-1]\{-1\}, y \in \overline{\mathcal{C}}(D(*1))[-1]\{-1\}\} \\
X_2 &= \{x + \partial y | x, y \in \overline{\mathcal{C}}(D(*000))\} \\
X_3 &= \{\alpha(x) + \partial\alpha(y) | x, y \in \overline{\mathcal{C}}(D(*110))[-2]\{-2\}\} \\
\overline{\mathcal{C}}(D') &= Y_1 \oplus Y_2 \oplus Y_3 \\
Y_1 &= \{x + \beta'(x) + y | x \in \overline{\mathcal{C}}(D'(*010))[-1]\{-1\}, y \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}\} \\
Y_2 &= \{x + \partial' y | x, y \in \overline{\mathcal{C}}(D'(*000))\} \\
Y_3 &= \{\alpha'(x) + \partial'\alpha'(y) | x, y \in \overline{\mathcal{C}}(D'(*110))[-2]\{-2\}\}
\end{aligned}
$$

On the cube on Figure (11), the complex is decomposed as the three subcomplexes X_1, X_2 and X_3. In the bottom of the cube, the decomposition does not change the bases for the cochain groups. Let's look at the top of the cube and see how this decomposition affects torsion. Note that the upper left cup is unchanged through the diagram. Hence, we can reduce the diagram since we are going to study locally the change of basis. Moreover, consider first the case when $\partial_{100 \to 110} = -m$.

Since $X_1^r = \{x + \beta(x) : x \in \overline{\mathcal{C}}(D(*100))[-1]\{-1\}^r\}$ then we get the basis:

$$
\begin{aligned}
(1 \otimes 1 &, & -1 \otimes 1), \\
(x \otimes 1 &, & -x \otimes 1), \\
(1 \otimes x &, & -x \otimes 1), \\
(x \otimes x &, & 0)
\end{aligned}
$$

which lies in $\overline{\mathcal{C}}(D(*100)^r \oplus \overline{\mathcal{C}}(D(*010))^r$.
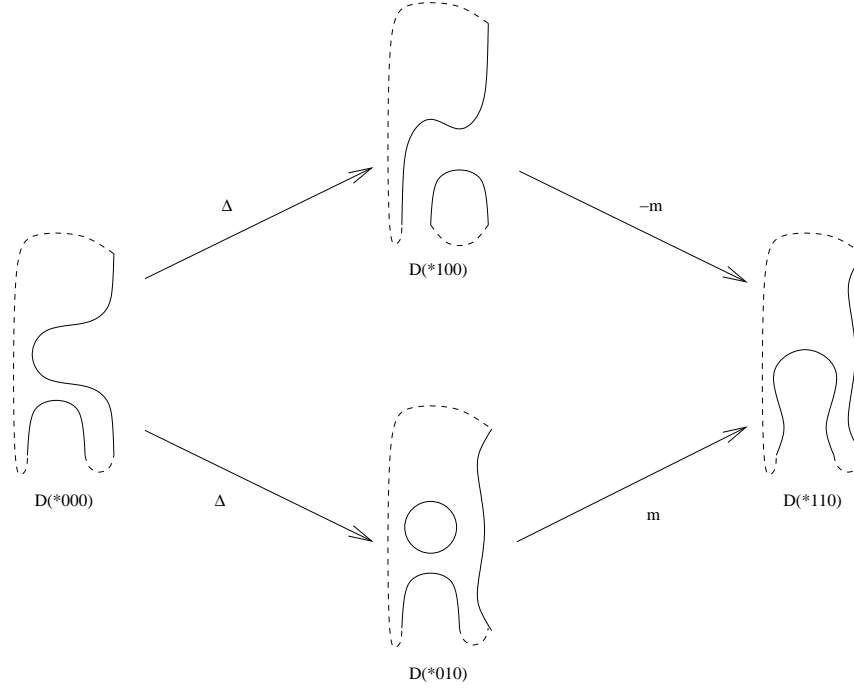
FIGURE 13. $\overline{\mathcal{C}}(D)$ for the case when $\partial_{100\to110} = -m$

From $X_2^r = \{x + \partial y | x \in \overline{\mathcal{C}}(D(*000))^r, y \in \overline{\mathcal{C}}(D(*000))^{r-1}\}$ we get the basis $\{1, x\}$ for $\overline{\mathcal{C}}(D(*000))^r$ and for the part that lies in $\overline{\mathcal{C}}(D(*100)^r \oplus \overline{\mathcal{C}}(D(*010))^r$, we get:

$$\{(1 \otimes x + x \otimes 1, 1 \otimes x + x \otimes 1), (x \otimes x, x \otimes x)\}$$

Finally, since $X_3^r = \{\alpha(x) + \partial\alpha(y) | x \in \overline{\mathcal{C}}(D(*110))[-2]\{-2\}^{r+1}, y \in \overline{\mathcal{C}}(D(*110))[-2]\{-2\}^r\}$, we get the basis $\{1, x\}$ for $\overline{\mathcal{C}}(D(*110))^r$ and for the part that lies in $\overline{\mathcal{C}}(D(*100))^r \oplus \overline{\mathcal{C}}(D(*010))^r$, we get:

$$\{(0, 1 \otimes 1), (0, x \otimes 1)\}$$

Therefore, the change of basis matrix for $\overline{\mathcal{C}}(D(*100)^i \oplus \overline{\mathcal{C}}(D(*010))^i$ between the bases described for X_1^r, X_2^r, and X_3^r and the distinguished basis is:

(15)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

which has determinant ± 1.

For the case when $\partial_{100\to110} = -\Delta$, the complex is shown.

Then X_1^r has basis:

$$\{(1, -1 \otimes x \otimes 1 - x \otimes 1 \otimes 1), (x, -x \otimes x \otimes 1)\}$$
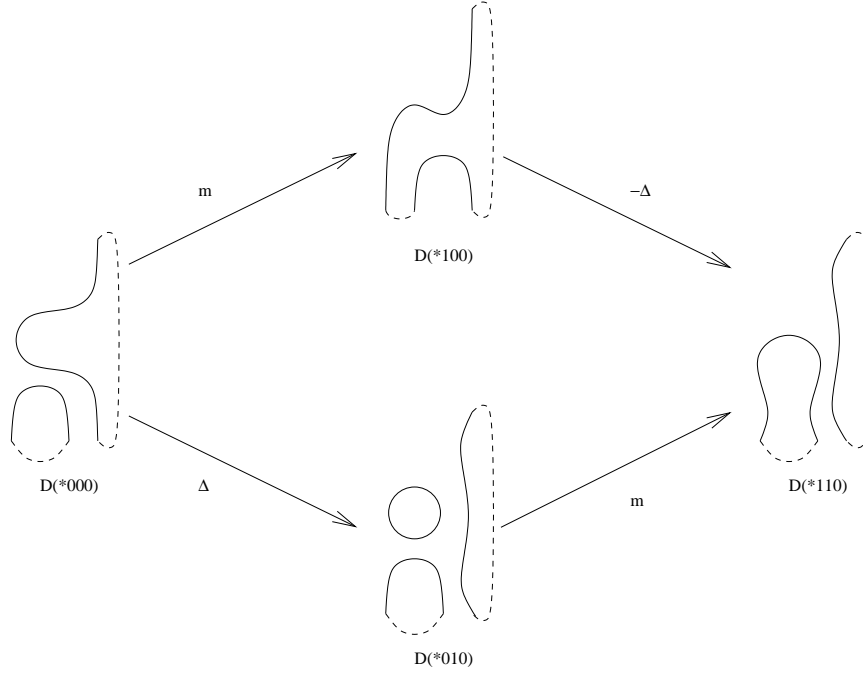
FIGURE 14. $\overline{\mathcal{C}}(D)$ for the case when $\partial_{100\to110} = -\Delta$

which lies in $\overline{\mathcal{C}}(D(*100)^r \oplus \overline{\mathcal{C}}(D(*010))^r$.

As before X_2^r, has the distinguished basis for $\overline{\mathcal{C}}(D(*000))^r$ and its contribution to $\overline{\mathcal{C}}(D(*100))^r \oplus \overline{\mathcal{C}}(D(*010))^r$ is:

$$
\begin{aligned}
(1 &, \quad 1 \otimes 1 \otimes x + x \otimes 1 \otimes 1) \\
(x &, \quad 1 \otimes x \otimes x + x \otimes x \otimes 1) \\
(x &, \quad x \otimes 1 \otimes x) \\
(0 &, \quad x \otimes x \otimes x)
\end{aligned}
$$

Also as before, X_3^r yields the distinguished basis for $\overline{\mathcal{C}}(D(*110))^r$ and for $\overline{\mathcal{C}}(D(*100)^r \oplus \overline{\mathcal{C}}(D(*010))^r$ it contributes:

$$
\begin{aligned}
(0 &, \quad 1 \otimes 1 \otimes 1) \\
(0 &, \quad x \otimes 1 \otimes 1) \\
(0 &, \quad 1 \otimes x \otimes 1) \\
(0 &, \quad x \otimes x \otimes 1)
\end{aligned}
$$

Therefore, the change of basis matrix for $\overline{\mathcal{C}}(D(*100)^r \oplus \overline{\mathcal{C}}(D(*010))^r$ between the bases described for X_1^r, X_2^r, and X_3^r and the distinguished basis is:

(16)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has determinant ± 1.

In both cases, the bases for the cochain groups $\overline{\mathcal{C}}(D(*000))^r$ and $\overline{\mathcal{C}}(D(*110))^r$ are the distinguished bases. For the cochain groups $\overline{\mathcal{C}}(D(*100)^r$ and $\overline{\mathcal{C}}(D(*010))^r$, the change of bases matrices have determinant ± 1. Hence, the bases for the cochain groups coming from the decomposition of the complex as the subcomplexes X_1, X_2, and X_3 are equivalent to the distinguished bases.

On the right side, Figure (12), the cube is also decomposed as three subcomplexes X_1, X_2, and X_3. Note that the decomposition does not change the basis used in the bottom of the cube. Let's look at the top of the cube after reducing the bottom right cup, which is constant throughout.
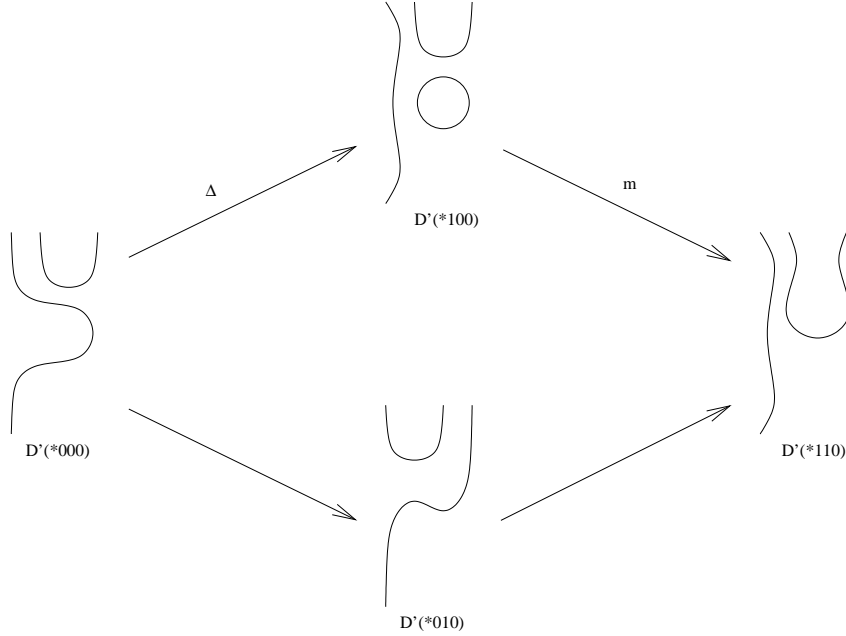

FIGURE 15. Reduced diagram for $\overline{\mathcal{C}}(D')$

After rotating each diagram by 180°, and interchanging the placement of the middle cochain groups in the direct sum, we obtain the same complex as the one

on the top of the left cube. The reader should look at the way the complex is decomposed and note the change of basis done for the the cube on Figure (11) is similar to the one done here.

Note that the acyclic (see [8]) subcomplexes:

$$\begin{aligned} X_2 &= \overline{\mathcal{C}}(D(*000)) \bigoplus \partial(\overline{\mathcal{C}}(D(*000))) \\ X_2' &= \overline{\mathcal{C}}(D'(*000)) \bigoplus \partial'(\overline{\mathcal{C}}(D'(*000))) \end{aligned}$$

are of the form $D \bigoplus \partial(D)$ and (since Δ has trivial kernel) they behave as the case for X_2 on the Reidemeister II move.

Furthermore, the acyclic (see [8]) subcomplexes:

$$\begin{aligned} X_3 &= \alpha(\overline{\mathcal{C}}(D(*110))) \bigoplus \partial(\alpha(\overline{\mathcal{C}}(D(*110)))) \\ X_3' &= \alpha'(\overline{\mathcal{C}}(D'(*110))) \bigoplus \partial'(\alpha'(\overline{\mathcal{C}}(D'(*110)))) \end{aligned}$$

behave exactly like the case for X_3 in the Reidemeister II move. (since $m(*, 1)$ is an onto map)

Furthermore: $\overline{\mathcal{C}}(D(*100))[-1]\{-1\}$ and $\overline{\mathcal{C}}(D'(*010))[-1]\{-1\}, \overline{\mathcal{C}}(D(*1))[-1]\{-1\}$ and $\overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$ are naturally isomorphic, and X_1 is isomorphic to Y_1 via

$$\rho : x + \beta(x) + y \longmapsto x + \beta'(x) + y .$$

and with the bases described the matrix representation of the quasi-isomorphism is the identity map for all the cochain groups and thus by (6) its Reidemeister torsion is 1.

Therefore, for all the Reidemeister moves, the quasi-isomorphisms have Reidemeister torsion 1. If two diagrams represent the same knot, then one can go from one to the other with a series of Reidemeister moves. The volume form is preserved by each move, giving an invariant volume form on the Khovanov invariant. The Khovanov chain complex decomposes into subcomplexes by polynomial degree. Since the acyclic subcomplexes used to demonstrate invariance have Reidemeister torsion 1, for acyclic subcomplexes corresponding to these polynomial degrees, Reidemeister torsion will be a number (there is no homology present), which will also be invariant for knots and links.

4.4. **Basis for homology groups.** Let's look at a special case of our construction by making the following choices for bases for the cohomology groups. Consider the cochain groups to have the same generators as explained before but being over $\mathbb{Z}$ instead of a field, then the cohomology groups (by the Fundamental Theorem of Abelian Groups) are of the form:

$$H(\mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z}_{p_i}^{n_i}$$

where the p_i's are prime numbers. Then, the quotient of $H(\mathbb{Z})$ by its torsion subgroup is free and it has the same rank (over $\mathbb{Z}$) as $H(\mathbb{F})$ (over $\mathbb{F}$). Over $\mathbb{Z}$ we can pick a basis for the free part of the homology groups and any two bases are related by a transition matrix in $Gl(n, \mathbb{Z})$. Since Khovanov homology is invariant over $\mathbb{Z}$, using this basis for the homology groups we obtain a numerical invariant for knots and links.

Computationally we can obtain such a basis by using the Smith normal form. Consider the short exact sequence:

$$0 \to B^{k-1} \xrightarrow{X^k} Z^k \xrightarrow{\pi} H^k \to 0 \tag{17}$$

where X_k is the matrix representation of the injection map with respect to a basis for B^{k-1} and a $\mathbb{Z}$-basis for $Z^k = \ker \partial^k$. Now consider the Smith normal form of X^k:

$$D = snf(X^k) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_l \end{pmatrix} \tag{18}$$

where $d_i = 1$ or d_i is an invariant factor of the matrix X_k. In particular, there are unitary matrices V and U such that $U \cdot X^k \cdot V = D$. In other words, U and V are change of basis matrices for B^{k-1} and Z^k, respectively, that make the injection map have matrix representation D. Zero rows of D will give a basis for the cohomology group after applying the change of basis matrix U.

Since we are only interested in a basis for the cohomology groups, we can restrict our study to the case where $U \cdot X^k = D$.

Theorem 3. *Let U and $U^{'}$ be unitary matrices such that $U \cdot X^k = D = U^{'} \cdot X^k$ where D and X^k are as specified above. Then the basis for the cohomology groups obtained from the zero rows of D after applying the change of bases matrices U and $U^{'}$ respectively are equivalent.*

Proof. By our hypothesis:

$$\begin{aligned} U \cdot X^k &= D = U^{'} \cdot X^k & \Leftrightarrow \\ U^{-1} \cdot D &= X^k = U^{'-1} \cdot D & \Leftrightarrow \\ U \cdot U^{'-1} \cdot D &= D \end{aligned} \tag{19}$$

From the structure of D, and column operations we can get that:

$$U \cdot U^{'-1} = \begin{pmatrix} M & 0 \\ * & Id \end{pmatrix} \sim \begin{pmatrix} M & 0 \\ 0 & Id \end{pmatrix} \tag{20}$$

Moreover, since U and $U^{'}$ are unitary:

$$\begin{aligned} 1 = 1 \ \cdot 1 = (\det U) \cdot (\det U^{'-1}) &= \det(U \cdot U^{'-1}) \\ &= \det(M) \cdot \det(Id) \\ &= \det(M) \end{aligned} \tag{21}$$

Since any two $\mathbb{Z}$-basis for Z^k have a change of basis matrix in $Gl(n, \mathbb{Z})$ and by (21), any two basis that we picked for our cohomology groups also have a change of basis $M \in Gl(n, \mathbb{Z})$, the bases for cohomology groups are equivalent, and hence do not change the torsion computation. $\square$

5. Examples

In 2004 Shumakovitch wrote KhoHo, a program for computing and studying Khovanov homology (see [12]). It is run using the PARI/GP calculator. For this reason, the routines for computing Reidemeister torsion for the Khovanov cochain complex were also written using PARI/GP code. The Khovanov cochain complex can be decompose by the grading in the vector space V, and in this way from each of these subcomplexes one obtains the corresponding coefficient for the term of that particular degree in the Jones polynomial. From KhoHo, by inputing a planar diagram representation for a given diagram, one can easily obtain the coboundary operators for each of the subcomplexes that form the Khovanov cochain complex.

5.1. **A detailed example.** The Hopf link contains generators in homological degrees -2, -1, and 0, and it is decomposed by polynomial degree into four subcomplexes of degrees 0, -2, -4, and -6. The output of the RTorsion routine returns a matrix.

```
 ? RTorsion(hopf)
%1 =

[1 1 1 1 "0"]

[1 -1 1 1 "-2"]

[1 -1 1 1 "-4"]

[-1 1 1 1 "-6"]
```

Each row correspond to a different subcomplex. In the last column of each row, the number inside the quotation marks indicates the polynomial degree for that particular subcomplex. The number in the previous column is the Reidemeister torsion for the subcomplex whose degree is specified in the last column. The rest of the entries in the matrix give the contribution to the torsion coming from each of the cochain groups contained in that particular subcomplex. For example, the subcomplex of polynomial degree -4 is

$$0 \to C_{-4}^{-2} \xrightarrow{\partial^{-2}} C_{-4}^{-1} \xrightarrow{\partial^{-1}} C_{-4}^0 \to 0$$

and $[h^{-2}\tilde{b}^{-2}/c^{-2}] = 1$, $[b^{-2}\tilde{b}^{-1}/c^{-1}] = -1$ (there is no homology here) and $[b^{-1}h^0/c^0] = 1$. Therefore $\tau_{-4} = \left| \frac{[b^{-2}\tilde{b}^{-1}/c^{-1}]}{[h^{-2}\tilde{b}^{-2}/c^{-2}] \cdot [b^{-1}h^0/c^0]} \right| = \left| \frac{-1}{1 \cdot 1} \right| = 1$. Note that these are the entries for the row corresponding to this subcomplex.

5.2. **Knots.** The names used for the knots in this table follow Rolfsen's table of knots [11]. The R-torsion for each knot is given as a table that contains the contribution to torsion from each cochain group (i.e. the determinant of the change of basis matrix between the basis obtained from the boundary operators and $\mathbb{Z}$-homology and the distinguished basis), and at the next to last column it contains the R-torsion for the subcomplex (i.e. the alternating product of the determinants of the change of basis matrices coming from each of the cochain groups) whose degree is specified in the last column.

Table 1: R-torsion for knots with up to 7 crossings	
knot	Reidemeister Torsion
knot3[1]	[1 1 1 1 1 ”-1”]
	[1 1 1 1 1 ”-3”]
	[1 -1 1 1 1 ”-5”]
	[1 -2 1 1 (1 /2) ”-7”]
	[1 1 1 1 1 ”-9”]
knot4[1]	[1 1 1 1 1 1 ”5”]
	[1 1 1 -1 2 (1 /2) ”3”]
	[1 -1 1 1 1 1 ”1”]
	[1 -1 -1 -1 1 1 ”-1”]
	[-1 -2 1 1 1 2 ”-3”]
	[1 1 1 1 1 1 ”-5”]
knot5[1]	[1 1 1 1 1 1 1 ”-3”]
	[1 1 -1 -1 1 1 1 ”-5”]
	[1 1 -1 1 1 1 1 ”-7”]
	[-1 1 1 2 1 1 (1 /2) ”-9”]
	[1 -1 1 1 1 1 1 ”-11”]
	[-1 2 1 1 1 1 (1 /2) ”-13”]
	[1 1 1 1 1 1 1 ”-15”]
knot5[2]	[1 1 1 1 1 1 1 ”-1”]
	[1 1 1 1 1 -1 1 ”-3”]
	[1 1 1 1 2 -1 2 ”-5”]
	[1 -1 -1 2 1 -1 (1 /2) ”-7”]
	[-1 1 -1 1 1 1 1 ”-9”]
	[1 -2 1 1 1 1 (1 /2) ”-11”]
	[1 1 1 1 1 1 1 ”-13”]
knot6[1]	[1 1 1 1 1 1 1 1 ”5”]
	[1 1 1 1 1 -1 -2 (1 /2) ”3”]
	[1 1 1 1 1 1 -1 1 ”1”]
	[1 1 -1 -1 -2 1 1 (1 /2) ”-1”]
	[1 -1 -1 -2 1 1 -1 2 ”-3”]
	[-1 1 1 1 1 -1 1 1 ”-5”]
	[1 2 -1 -1 1 1 1 2 ”-7”]
	[1 1 1 1 1 1 1 1 ”-9”]
knot6[2]	[1 1 1 1 1 1 1 1 ”3”]
	[1 1 1 1 1 1 -2 (1 /2) ”1”]
	[1 -1 -1 -1 -1 1 -1 1 ”-1”]
	[1 -1 -1 1 -2 -1 1 (1 /2) ”-3”]
	[-1 -1 1 -2 1 1 1 2 ”-5”]
	[-1 -1 -2 1 1 1 1 (1 /2) ”-7”]
	[-1 2 1 1 1 1 1 2 ”-9”]
	[1 1 1 1 1 1 1 1 ”-11”]

Table 1 – continued	
knot6[3]	[1 1 1 1 1 1 1 1 ”7”]
	[1 1 1 1 1 1 -2 2 ”5”]
	[1 1 1 1 1 2 1 (1 /2) ”3”]
	[1 1 1 -1 -2 1 -1 2 ”1”]
	[1 1 1 2 -1 -1 1 (1 /2) ”-1”]
	[1 -1 -2 -1 1 1 1 2 ”-3”]
	[1 -2 1 1 1 1 1 (1 /2) ”-5”]
	[1 1 1 1 1 1 1 1 ”-7”]
knot7[1]	[1 1 1 1 1 1 1 1 1 ”-5”]
	[1 1 -1 -1 -1 -1 1 1 1 ”-7”]
	[1 -1 -1 -1 1 -1 1 1 1 ”-9”]
	[1 -1 -1 -1 1 2 1 1 (1 /2) ”-11”]
	[-1 1 -1 1 1 1 1 1 1 ”-13”]
	[-1 -1 -1 -2 1 1 1 1 (1 /2) ”-15”]
	[-1 -1 1 1 1 1 1 1 1 ”-17”]
	[-1 -2 1 1 1 1 1 1 (1 /2) ”-19”]
	[1 1 1 1 1 1 1 1 1 ”-21”]
knot7[2]	[1 1 1 1 1 1 1 1 1 ”-1”]
	[1 1 1 1 1 1 1 1 1 ”-3”]
	[1 1 1 1 -1 -1 2 -1 2 ”-5”]
	[1 1 1 1 1 -2 1 -1 (1 /2) ”-7”]
	[1 1 1 1 2 -1 -1 1 2 ”-9”]
	[1 -1 1 2 -1 -1 1 -1 (1 /2) ”-11”]
	[-1 -1 -1 1 -1 -1 1 1 1 ”-13”]
	[1 -2 -1 -1 1 1 1 1 (1 /2) ”-15”]
	[1 1 1 1 1 1 1 1 1 ”-17”]
knot7[3]	[1 1 1 1 1 1 1 1 1 ”19”]
	[1 1 1 1 1 -1 1 2 2 ”17”]
	[1 1 1 -1 -1 1 -1 -1 1 ”15”]
	[1 1 -1 -1 1 4 -1 -1 4 ”13”]
	[1 1 -1 -1 -2 -1 -1 -1 (1 /2) ”11”]
	[-1 1 1 2 -1 -1 1 1 2 ”9”]
	[-1 1 -2 -1 1 1 1 1 (1 /2) ”7”]
	[1 -1 -1 -1 1 1 1 1 1 ”5”]
	[1 1 1 1 1 1 1 1 1 ”3”]
knot7[4]	[1 1 1 1 1 1 1 1 1 ”17”]
	[1 1 1 1 1 -1 -1 2 2 ”15”]
	[1 1 -1 -1 1 1 -1 -1 1 ”13”]
	[1 1 -1 1 -1 4 1 1 4 ”11”]
	[-1 1 1 1 -2 -1 1 1 (1 /2) ”9”]
	[1 1 -1 2 1 1 1 1 2 ”7”]
	[1 -1 4 1 1 1 1 1 (1 /4) ”5”]
	[1 1 1 1 1 1 1 1 1 ”3”]
	[1 1 1 1 1 1 1 1 1 ”1”]

Table 1 – continued	
knot7[5]	[1 1 1 1 1 1 1 1 1 ”-3”]
	[1 1 1 1 1 -1 1 -1 1 ”-5”]
	[1 1 1 1 1 -1 2 -1 2 ”-7”]
	[1 -1 1 1 -1 -4 1 -1 (1 /4) ”-9”]
	[1 -1 1 1 -2 1 1 1 2 ”-11”]
	[1 -1 1 4 1 1 1 1 (1 /4) ”-13”]
	[1 1 -2 1 1 1 1 1 2 ”-15”]
	[1 2 1 1 1 1 1 1 (1 /2) ”-17”]
	[1 1 1 1 1 1 1 1 1 ”-19”]
knot7[6]	[1 1 1 1 1 1 1 1 1 ”3”]
	[1 1 1 1 1 1 1 -2 (1 /2) ”1”]
	[1 1 1 -1 -1 -1 2 1 2 ”-1”]
	[1 -1 -1 1 1 -2 1 -1 (1 /2) ”-3”]
	[1 1 1 -1 4 1 1 1 4 ”-5”]
	[1 -1 -1 -4 1 1 1 1 (1 /4) ”-7”]
	[-1 1 -2 1 1 1 1 1 2 ”-9”]
	[1 -2 1 1 1 1 1 1 (1 /2) ”-11”]
	[1 1 1 1 1 1 1 1 1 ”-13”]
knot7[7]	[1 1 1 1 1 1 1 1 1 ”9”]
	[1 1 1 1 1 1 -1 -2 (1 /2) ”7”]
	[1 1 1 1 1 1 -2 -1 2 ”5”]
	[1 1 -1 -1 1 -4 -1 -1 (1 /4) ”3”]
	[-1 -1 1 -1 4 -1 1 1 4 ”1”]
	[-1 -1 -1 -2 -1 1 1 1 (1 /2) ”-1”]
	[-1 1 -4 -1 1 1 1 1 4 ”-3”]
	[-1 2 1 1 1 1 1 1 (1 /2) ”-5”]
	[1 1 1 1 1 1 1 1 1 ”-7”]

5.3. **Links.** The names used for the links in this table follow the Thistlethwaite link table [5]. (See the previous section to understand the table.)

Table 2: R-torsion for links with up to 7 crossings	
link	Reidemeister Torsion
link2a[1]	[1 1 1 1 ”0”]
	[1 -1 1 1 ”-2”]
	[-1 -1 1 1 ”-4”]
	[1 1 1 1 ”-6”]
link4a[1]	[1 1 1 1 1 1 ”0”]
	[1 1 1 1 -1 1 ”-2”]
	[1 1 -1 2 -1 2 ”-4”]
	[1 -1 1 -1 -1 1 ”-6”]
	[-1 -1 -1 -1 1 1 ”-8”]
	[1 1 1 1 1 1 ”-10”]

Table 2 – continued	
link5a[1]	[1 1 1 1 1 1 1 ”4”]
	[1 1 1 1 1 -2 (1 /2) ”2”]
	[1 1 1 1 1 1 1 1 ”0”]
	[1 -1 1 -1 1 1 1 ”-2”]
	[1 -1 -2 1 1 1 2 ”-4”]
	[-1 -2 1 1 1 1 (1 /2) ”-6”]
	[1 1 1 1 1 1 1 ”-8”]
link6a[1]	[1 1 1 1 1 1 1 1 1 ”4”]
	[1 1 1 1 1 1 2 (1 /2) ”2”]
	[1 1 1 -1 1 -2 -1 2 ”0”]
	[1 1 -1 -1 1 1 -1 1 ”-2”]
	[-1 -1 -1 -4 -1 -1 1 4 ”-4”]
	[-1 1 -2 1 1 1 1 (1 /2) ”-6”]
	[1 1 1 1 1 1 1 1 ”-8”]
	[1 1 1 1 1 1 1 1 ”-10”]
link6a[2]	[1 1 1 1 1 1 1 1 ”-2”]
	[1 1 1 1 1 1 -1 1 ”-4”]
	[1 1 1 -1 -1 2 -1 2 ”-6”]
	[1 1 1 1 -2 -1 -1 (1 /2) ”-8”]
	[-1 1 -1 -2 -1 -1 1 2 ”-10”]
	[1 1 -2 1 1 1 1 (1 /2) ”-12”]
	[-1 -1 1 1 1 1 1 1 ”-14”]
	[1 1 1 1 1 1 1 1 ”-16”]
link6a[3]	[1 1 1 1 1 1 1 1 ”-4”]
	[1 -1 -1 1 -1 -1 1 1 ”-6”]
	[1 -1 -1 1 -1 -1 1 1 ”-8”]
	[-1 -1 1 1 2 1 1 (1 /2) ”-10”]
	[1 -1 1 -1 1 1 1 1 ”-12”]
	[-1 -1 -2 1 1 1 1 (1 /2) ”-14”]
	[-1 1 1 1 1 1 1 1 ”-16”]
	[1 1 1 1 1 1 1 1 ”-18”]
link6a[4]	[1 1 1 1 1 1 1 1 ”7”]
	[1 1 1 1 1 1 -2 2 ”5”]
	[1 1 1 1 1 4 -1 (1 /4) ”3”]
	[1 -1 -1 -1 1 1 -1 1 ”1”]
	[1 1 -1 1 -1 -1 1 1 ”-1”]
	[1 -1 -4 -1 1 1 1 4 ”-3”]
	[1 -2 1 1 1 1 1 (1 /2) ”-5”]
	[1 1 1 1 1 1 1 1 ”-7”]
link6a[5]	[1 1 1 1 1 1 1 1 ”-1”]
	[1 1 1 1 1 1 1 1 1 ”-3”]
	[1 1 1 1 -1 -4 -1 4 ”-5”]
	[1 1 1 1 2 -1 1 (1 /2) ”-7”]
	[1 -1 -1 -1 -1 -1 -1 1 ”-9”]
	[1 -1 -1 1 -1 -1 1 1 ”-11”]
	[1 2 1 1 1 1 1 2 ”-13”]
	[1 1 1 1 1 1 1 1 ”-15”]

Table 2 – continued	
link6n[1]	[1 1 1 1 1 1 1 1 ”9”]
	[1 1 1 1 -1 1 1 1 ”7”]
	[1 1 -1 -1 -1 -1 -1 1 ”5”]
	[1 1 1 -1 -2 -1 1 (1 /2) ”3”]
	[1 -1 -1 1 1 1 1 1 ”1”]
	[1 -1 1 1 1 1 1 1 ”-1”]
link7a[1]	[1 1 1 1 1 1 1 1 1 ”10”]
	[1 1 1 1 1 1 1 2 (1 /2) ”8”]
	[1 1 1 1 1 -1 4 -1 4 ”6”]
	[1 1 1 1 1 -4 -1 1 (1 /4) ”4”]
	[1 1 1 1 4 -1 1 -1 4 ”2”]
	[-1 -1 -1 -2 1 -1 1 1 (1 /2) ”0”]
	[-1 1 -4 -1 1 1 1 4 ”-2”]
	[-1 2 1 1 1 1 1 1 (1 /2) ”-4”]
	[1 1 1 1 1 1 1 1 1 ”-6”]
link7a[2]	[1 1 1 1 1 1 1 1 1 ”-2”]
	[1 1 1 1 1 1 1 1 1 ”-4”]
	[1 1 1 1 1 -1 4 1 4 ”-6”]
	[1 1 1 1 -1 4 -1 1 (1 /4) ”-8”]
	[1 -1 1 1 -2 -1 -1 -1 2 ”-10”]
	[-1 -1 1 4 -1 -1 1 1 (1 /4) ”-12”]
	[-1 -1 2 -1 1 1 1 1 2 ”-14”]
	[1 -2 1 1 1 1 1 1 (1 /2) ”-16”]
	[1 1 1 1 1 1 1 1 1 ”-18”]
link7a[3]	[1 1 1 1 1 1 1 1 1 ”14”]
	[1 1 1 1 1 1 1 2 2 ”12”]
	[1 1 1 1 1 1 2 -1 (1 /2) ”10”]
	[1 1 1 1 -1 -4 1 -1 4 ”8”]
	[1 1 1 -1 -2 1 1 1 (1 /2) ”6”]
	[1 1 1 2 -1 -1 1 -1 2 ”4”]
	[1 -1 -1 1 -1 -1 1 1 1 ”2”]
	[1 -2 1 1 1 1 1 1 2 ”0”]
	[1 1 1 1 1 1 1 1 1 ”-2”]
link7a[4]	[1 1 1 1 1 1 1 1 1 ”12”]
	[1 1 1 1 1 1 -1 -2 2 ”10”]
	[1 1 1 -1 -1 1 -2 -1 (1 /2) ”8”]
	[1 -1 1 -1 1 2 1 -1 2 ”6”]
	[-1 -1 1 1 4 -1 1 1 (1 /4) ”4”]
	[1 1 -1 -2 1 1 1 1 2 ”2”]
	[-1 -1 -1 -1 1 1 1 1 1 ”0”]
	[-1 -2 1 1 1 1 1 1 2 ”-2”]
	[1 1 1 1 1 1 1 1 1 ”-4”]

Table 2 – continued	
link7a[5]	[1 1 1 1 1 1 1 1 1 ”4”]
	[1 1 1 1 1 1 1 2 (1 /2) ”2”]
	[1 1 1 1 1 1 -2 -1 2 ”0”]
	[1 1 1 -1 -1 -2 1 1 (1 /2) ”-2”]
	[1 1 1 1 -4 -1 1 -1 4 ”-4”]
	[1 1 1 -2 1 1 1 1 (1 /2) ”-6”]
	[1 1 2 1 1 1 1 1 2 ”-8”]
	[-1 -2 1 1 1 1 1 1 (1 /2) ”-10”]
	[1 1 1 1 1 1 1 1 1 ”-12”]
link7a[6]	[1 1 1 1 1 1 1 1 1 ”14”]
	[1 1 1 1 1 1 1 -1 2 2 ”12”]
	[1 1 1 1 1 1 2 -1 (1 /2) ”10”]
	[1 1 1 -1 1 -2 -1 1 2 ”8”]
	[1 1 1 -1 2 1 -1 1 (1 /2) ”6”]
	[1 1 1 2 1 -1 1 -1 2 ”4”]
	[1 -1 1 -1 -1 -1 1 1 1 ”2”]
	[1 2 1 -1 1 1 1 1 2 ”0”]
	[1 1 1 1 1 1 1 1 1 ”-2”]
link7a[7]	[1 1 1 1 1 1 1 1 1 ”7”]
	[1 1 1 1 1 1 1 -1 2 2 ”5”]
	[1 1 1 1 1 -1 4 -1 (1 /4) ”3”]
	[1 1 1 1 -1 2 1 1 2 ”1”]
	[1 1 -1 1 1 1 1 -1 1 ”-1”]
	[1 -1 -1 -8 -1 -1 1 1 8 ”-3”]
	[-1 -1 -2 1 1 1 1 (1 /2) ”-5”]
	[-1 -1 1 1 1 1 1 1 1 ”-7”]
	[1 1 1 1 1 1 1 1 1 ”-9”]
link7n[1]	[1 1 1 1 1 1 1 1 1 ”-4”]
	[1 1 1 1 -1 1 -1 -1 1 ”-6”]
	[1 1 -1 1 1 1 -1 1 1 ”-8”]
	[1 -1 1 -1 1 2 1 1 (1 /2) ”-10”]
	[1 1 1 1 -1 -1 1 1 1 ”-12”]
	[-1 -1 -1 2 -1 1 1 1 (1 /2) ”-14”]
	[1 -1 -1 1 1 1 1 1 1 ”-16”]
link7n[2]	[1 1 1 1 1 -1 1 1 1 ”0”]
	[1 1 -1 1 -1 1 -1 1 1 ”-2”]
	[1 -1 -1 -1 -2 1 -1 1 2 ”-4”]
	[-1 1 1 -2 -1 1 1 1 (1 /2) ”-6”]
	[-1 -1 -1 -1 1 1 1 1 1 ”-8”]
	[-1 -2 -1 1 1 1 1 1 (1 /2) ”-10”]
	[1 1 1 1 1 1 1 1 1 ”-12”]

5.4. **Remarks from the computations.** Consider the trefoil knot (knot 3_1 in the Rolfsen table [11]). From the Knot Atlas [2] we obtain the following information about the Khovanov homology groups for this knot, when homology is computed over $\mathbb{Z}$:

Table 3: Homology groups for the trefoil	
Subcomplex	Homology
-1	$\mathbb{Z}$
-3	$\mathbb{Z}$
-5	$\mathbb{Z}$
-7	$\mathbb{Z}_2$
-9	$\mathbb{Z}$

The subcomplex of degree -7, has homology $\mathbb{Z}_2$, a torsion group and it contains no free components. In fact over $\mathbb{Q}$, this subcomplex is acyclic but its Reidemeister torsion (see Section 5.2) is given by:

```
[1 1 1 1 1 "-1"]
[1 1 1 1 1 "-3"]
[1 -1 1 1 1 "-5"]
[1 -2 1 1 (1 /2) "-7"]
[-1 1 1 1 1 "-9"]
```

Note, that the acyclic subcomplex of degree -7, recovers the algebraic torsion exhibited when the homology is calculated over $\mathbb{Z}$. In fact, the same holds true for all the knots and links in Sections 5.2 and 5.3, due to the choices made for the homology groups.

References

[1] D. Bar-Natan, On Khovanov's categorification of the Jones Polynomial, Algebr. and Geom. Topology 2:16, (electronic), p. 337-370, 2002

[2] D. Bar-Natan, The knot atlas, http: //katlas.math.toronto.edu/

[3] J.W. Chung and X.S. Lin, Torsion of quasi-isomorphisms, arXiv:math.AT/0608459 v2, 2006

[4] W. Franz, Ueber die Torsion einer Ueberdeckung, J. Reine Angew. Math. 173, 1935

[5] M. Thistlethwaite, The Thistlethwaite link table, http://katlas.math.toronto.edu/wiki/The_Thistlethwaite_Link_Table, 1991

[6] V. Jones, A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc. vol. 12, n.1, p. 103-111, 1985

[7] L. H. Kauffman, State models and the Jones polynomial, Topology 26 no. 3, p. 395–407, 1987

[8] M. Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101 (3), p. 359-426, 1999

[9] E. Lee, An endomorphism of the Khovanov invariant, Adv. Math. 197 MR2173845, p. 554-586, 2005

[10] K. Reidemeister, Homotopieringe und Linseräume, Hamburger Abhandlungen 11, 1935

[11] D. Rolfsen, Knots and Links, AMS Chelsea Publishing, 2003

[12] A. Shumakovitch, KhoHo version 0.9.3.5, http://www.geometrie.ch/KhoHo,

[13] M. Spivak, Calculus on Manifolds, Westview Publisher, 1965

[14] V. Turaev, Introduction to Combinatorial Torsion, Birkhäuser Publisher, 2001

E-mail address: juorna@gmail.com

Department of Mathematics, Hylan Building, University of Rochester, Rochester, NY, 14627